LEFSCHETZ PROPERTIES IN ALGEBRA, GEOMETRY, TOPOLOGY AND COMBINATORICS

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Abstracts

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Bernstein–Sato polynomials of projective line arrangements

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The roots of Bernstein–Sato polynomial of a hypersurface manage to simultaneously contain most classical singularity invariants. Alas, computing these roots is mostly infeasible. For hyperplane arrangements in $\mathbb{C}^3$, one can hope the roots of its Bernstein–Sato polynomial are combinatorially determined. Alas, things are more subtle. Walther demonstrated two arrangements with the same intersection lattice but whose respective Bernstein–Sato polynomials differ by exactly one root.

We will show this is the only pathology possible. For arrangements in $\mathbb{C}^3$, we prove that all but one root are (easily) combinatorially determined. We also give several equivalent criterion for the outlier, $-2+(2/\deg)$, to in fact be a root of the Bernstein–Sato polynomial. These involve local cohomology data of the Milnor algebra and the non-formality of the arrangement. This is an application of a study of Bernstein-Sato polynomials for a larger class of $\mathbb{C}^3$ divisors than just arrangements.

Given the audience, we will sketch what the Bernstein–Sato polynomial is and why one should care. We will gesture at the duality type argument used in the proof and the crucial role Lefschetz properties play.

The Weak Lefschetz property for Artinian Gorenstein algebras with low Sperner number

Mats Boij

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For Artinian Gorenstein algebras in codimension four and higher, it is well known that the Weak Lefschetz Property (WLP) does not need to hold. For Gorenstein algebras in codimension three, it is still open whether all Artinian Gorenstein algebras satisfy the WLP when the socle degree and the Sperner number are both higher than six.

In recent joint work with Juan Migliore, Rosa Maria Miró-Roig and Uwe Nagel, we show that all Artinian Gorenstein algebras with socle degree d and Sperner number at most $d+1$ satisfy the WLP, independent of the codimension. This is a sharp bound in general since there are examples of Artinian Gorenstein alebras with socle degree d and Sperner number $d+1$ that don't satisfy the WLP for all $d \geq 3$.

The strong Lefschetz property of certain modules over Clements-Lindström rings

Bek Chase

Purdue University, United States

Much work has been done in studying the Lefschetz properties for Artinian graded algebras over a field. Less is known about the properties for graded modules over a polynomial ring, even in few variables. In this talk, I will show how the combinatorial Lindström-Gessel-Viennot Lemma can be used to prove $k[x, y]$-modules have the strong Lefschetz property in characteristic zero. As an application, I will discuss how these results for modules can be used to prove certain codimension three algebras have the strong Lefschetz property.

Weddle loci and the structure of projections

Luca Chiantini

Università di Siena, Italy

I will shortly illustrate how the procedure introduced with the construction of Weddle loci, which is connected with the determination of the Lefschetz locus of certain algebras, can be generalized in order to describe the notion of projection of linear systems of hypersurfaces.

Lefschetz type properties of quasi-projective varieties

Ananyo Dan

CUNEF Universidad, Spain

The talk will cover two interesting results due to Lefschetz. The first is the classical Lefschetz hyperplane section theorem, which is known to hold for smooth, projective varieties. We observe that the theorem can fail if we remove just a single point from a projective variety. The second part of the talk will focus on the Noether-Lefschetz theorem. This deals with the case not covered by the Lefschetz hyperplane theorem. I will recall classical density results and geometry of the Noether-Lefschetz locus. I will end with a conjecture of Harris on this locus.

On the Weak Lefschetz Property for certain ideals generated by powers of linear forms

Giuseppe Favacchio

Università degli Studi di Palermo, Italy

Ideals $I \subseteq R = k[\mathbb{P}^n]$ generated by powers of linear forms arise, via Macaulay duality, from sets of fat points $X \subseteq \mathbb{P}^n$. Properties of R/I are connected to the geometry of the corresponding fat points. When the linear forms are general, many authors have studied the question of whether or not R/I has the Weak Lefschetz Property (WLP). We study this question instead for ideals coming from a family of sets of points called grids. We give a complete answer in the case of uniform powers of linear forms coming from square grids, and we give a conjecture and approach for the case of nonsquare grids. In the cases where WLP holds, we also describe the non-Lefschetz locus.

On higher Jacobians, Laplace equations and Lefschetz properties

Rodrigo Gondim

UFRPE, Brazil

Let A be a standard graded Artinian $\mathbb{K}$-algebra over an algebraically closed field of characteristic zero. We use apolarity to construct, for each degree k, a projective variety whose osculating defect in degree s is equivalent to the non maximality of the rank of the multiplication map for a power of a general linear form $\times L^{k-s} : A_s \to A_k$.

This construction allows us to re-obtain some of the foundational theorems in the field. The results presented in this work provide new insights on the geometry of monomial Togliatti systems, and offer a geometric interpretation of the vanishing of higher order Hessians.

Recent results on geproci sets

Brian Harbourne

University of Nebraska, United States

The notion of a greproci set is only a few years old, growing out of work done at the Levico Lefschetz meeting in 2019. It is based on a new model for classifying point sets in projective space, motivated by inverse scattering, namely: classify point sets according to the behavior of the image of the set under projection from a general point to a hyperplane. This provides many still open avenues for research. I will focus on the property that the projection is a complete intersection and discuss some recent results.

Lefschetz properties of squarefree monomial ideals via Rees algebras

Thiago Holleben

Dalhousie University, Canada

In this talk we show how the theory of Rees algebras can be used to find interesting examples in the theory of Lefschetz properties. We also explore the consequences of known results from Lefschetz properties to the Rees algebras of squarefree monomial ideals.

JORDAN TYPE FOR ARTINIAN GORENSTEIN ALGEBRAS, AND PUNCTUAL SCHEMES

Tony Iarrobino

Northeastern University, United States

Let R be the polynomial ring over an algebraically closed field k, with maximum ideal $\mathfrak{m}$, and consider length n Artinian quotients $A = R/I$ of R. Let $\ell \in \mathfrak{m}$, then the multiplication map $m_\ell : A \to A$ is nilpotent, so has Jordan normal form determined by a partition $P_{A,\ell}$ of n, the *Jordan type* of the pair (A, ℓ). The set $\mathcal{P}_A = \{P_{A,\ell}, \ell \in \mathfrak{m}\}$ is an invariant of A. For each ℓ, A has a decomposition as a direct sum of *strings*: simple $\mathsf{k}[\ell]$ modules. When A is graded, the *Jordan degree type* gives the initial degrees of the strings. We consider these invariants and their properties for codimension three graded Artinian Gorenstein algebras of almost constant Hilbert function $T = (1, 3, s, \ldots, s, 3, 1)$ when $s \leq 6$: we determine the JDT given T for $s \leq 5$ and delimit the possibilities for $s = 6$. The families $\mathrm{Gor}(T)$ parametrizing these are closely related to length-s punctual schemes $\mathfrak{Z}$ in $\mathbb{P}^2$. A key tool is the connection between these punctual schemes and the rank matrices, JDT matrices introduced by N. Altafi. The talk is based on a joint work with N. Abdallah, N. Altafi and J. Yaméogo.

On minimal Gorenstein Hilbert functions

Giovanna Ilardi

Dipartimento di Matematica Università degli studi di Napoli "Federico II", Italy

We study Artinian Gorenstein (graded) algebras with minimal Hilbert function. We conjecture that a class of Artinian Gorenstein algebras called full Perazzo algebras always have minimal Hilbert function, fixing codimension and length.

We prove the conjecture in length four and five, in low codimension. We also prove the conjecture for a particular subclass of algebras that occurs in every length and certain codimensions.

Finally we give a new proof of part of a known result about the asymptotic behavior of the minimum entry of a Gorenstein Hilbert function. The talk is based on a joint work with L. Bezerra, R. Gondim, and G. Zappalà.

Some graphs giving algebras with the strong Lefschetz property

Filip Jonsson Kling

Stockholm University, Sweden

Given a simple graph G, one can construct an Artinian algebra $A(G)$ associated to it by considering the polynomial ring with variables labelled by the vertices of the graph modulo the edge ideal of the graph and the squares of all of the variables. In a recent work of Altafi and Lundqvist, they gave explicit bounds on the number of vertices and edges of a graph G which forces the algebra $A(G)$ to have the weak Lefschetz property (WLP) and moreover showed that in all other cases, there are graphs which fail to have the WLP.

In this talk, I will answer a converse question to that of Altafi and Lundqvist. Namely, given some specified number of vertices and edges of a simple graph, is it possible to construct a graph G with those specifications such that $A(G)$ has the WLP, or perhaps even the strong Lefschetz property (SLP)? The answer to both those questions turns out to be yes! I will describe a family of such graphs G such that $A(G)$ has the SLP and talk about some interesting properties of their associated Artinian algebras, especially some results on their Hilbert series that might be of independent interest.

Commuting Jordan Types

Leila Khatami

Union College, United States

In this talk, I will provide a survey of the recent progress in problems concerning the maximum commuting nilpotent orbit that intersects with the centralizer of a given nilpotent matrix.

A Gröbner basis argument for $K[x_1, \ldots, x_n]/(x_1^2, \ldots, x_n^2)$ having the SLP

Samuel Lundqvist

Stockholm University, Sweden

We determine the Gröbner basis for the ideal $(x_1^2, \ldots, x_n^2, (x_1 + \cdots + x_n)^k)$, which turns out to have an interesting structure. As a side effect we obtain a new proof of the fact that $K[x_1, \ldots, x_n]/(x_1^2, \ldots, x_n^2)$ has the SLP. Here K denotes a field of characteristic zero. This is a joint work with Eduardo Sáenz De Cabezón, Fatemeh Mohammadi, Filip Jonsson Kling and Matthias Orth.

Weighted path matrices and Almkvist's conjecture in codimension two

Chris McDaniel

Endicott College, United States

In the 1980s, G. Almkvist formulated a conjecture regarding the unimodality of the generating functions of a certain two parameter family of integer partitions. In a 2019 paper with S. Chen, A. Iarrobino, and P. Macias Marques, we discovered (or perhaps rediscovered?) that this family of generating functions are also the Hilbert functions of a certain two parameter family of (nonstandard) graded Artinian complete intersection algebras, and we further conjectured that its unimodality comes from Lefschetz properties of these algebras.

In recent work with N. Abdallah, we verified this conjecture in the codimension two case using weighted path matrices. I will discuss these results, as well as some interesting connections to Catalan numbers and NE lattice paths.

Artinian Cox-Gorenstein Algebras

William Montoya

University of Ferrara, Italy

In this talk mainly I will introduce non-standard graded algebras having the Poincaré duality and the notion of Lefschetz properties.

Firstly, I will show the equivalence between Toric varieties and some G-graded algebras, where G is a finitely generated abelian group, possibly with torsion.

Secondly, I will present a generalization of Macaulay-Matlis duality, introduce Lefschetz properties and a Hessian criteria in this G-graded setup. Then I will establish the toric Macaulay duality theorem.

Finally, if time allows me, I will indicate a connection between Noether-Lefschetz theory in toric varieties and Artinian Cox-Gorenstein Algebras. This is joint work with Ugo Bruzzo, Rodrigo Gondim and Rafael Holanda.

Lefschetz properties of some level algebras arising from graphs

Lisa Nicklasson

Mälardalen University, Sweden

A simple graph gives rise to an Artinian monomial ideal by taking the edge ideal and adding the squares of the variables. If the graph is a so called *whiskered graph*, then the Artinian algebra produced is level.

In this talk I will explain this construction, and then discuss the Lefschetz properties (or lack thereof) for algebras of this type. The talk is based on a joint work S. Cooper, S. Faridi, T. Holleben, and A. Van Tuyl.

Monomial Togliatti systems

Aline Vilela Andrade

Universidade Federal de Minas Gerais/Università di Trieste, Brazil

In this talk, we delve into Monomial Togliatti Systems. We will provide an overview of the foundational concepts, recent advancements, and related open problems.

Binary Trees of complete intersections

Junzo Watanabe

Tokai University, Japan

Let $R = K[x_1, x_2, \ldots, x_n]$. Let $\{f_k\} \subset R$ be a sequence of homogeneous polynomials with $\deg f_k = k$. They satisfy a generalized Newton identity if

$$w_k e_k = \sum_{j=1}^{k-1} (-1)^{j-1} f_{k-j} e_j, \quad k \leq n,$$

$$f_k = \sum_{j=1}^{n} (-1)^{j-1} f_{k-j} e_j, \quad k > n,$$

for some $w_i \in K$, $w_1 \cdots w_n \neq 0$. (e_k is the elementary symmetric polynomial of degree k.)

We show the following. Let $n = 3$. If $I = (f_d, f_{d+1}, f_{d+2}) \subset R$ is a complete intersection generated by homogeneous elements of degrees $d, d+1, d+2$, and if $(I : x_3^k)$ is a complete intersection for all $k \geq 0$, then $R/(I : x_3^k)$ have the SLP for all $k \geq 0$. Furthermore we prove that if $I = (f_d, f_{d+1}, f_{d+2})$ and if $\{f_k\}$ satisfy a generalized Newton identity, then $(I : x_3^k)$ is a complete intersection for all $k \geq 0$. As a consequence $R/(I : x_3^k)$ has the SLP for all $k \geq 0$.

Let $R = K[x_1, \ldots, x_n]$. Let $q \in K$ be an arbitrary element. Suppose $\{f_k\}$ satisfy the Newton identity with $w_i = \frac{1-q^i}{1-q}$, then $(f_d, f_{d+1} \ldots, f_{d+n-1}) : x_n^k$ is a complete intersection for all $k \geq 0$. We can use this fact to prove that $A = R/(I : x_n^k)$ has the SLP for all $k \geq 0$.